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DERIVATION OF GEOMETRIC STIFFNESS AND MASS MATRICES FOR FINITE ELEMENT HYBRID MODELS†

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Abstract—An unifying approach in deriving the geometric stiffness and mass matrices for finite element hybrid models is presented. The variational formulation is based on a modified Reissner Principle. Numerical verification is illustrated through a simple beam example. The element interchangeability in the finite element method and an alternative way of deriving the loading vector for the hybrid stress model are also discussed.

1. INTRODUCTION

Since Pian first established the assumed stress hybrid finite element model and derived the corresponding element stiffness matrix in 1964[1], in a series of papers [2–5] he and his associates have shown that such hybrid models not only provide the flexibility and easiness in fulfilling the compatibility condition of the finite element method for plates and shells but are also mathematically sound and highly accurate. In the meantime, many authors have also confirmed the usefulness of such a model and have adopted it for many applications, in solid mechanics as well as in fluid mechanics (e.g. Refs.[6–9], and many others) Pian has provided a review of the development of the hybrid models in[10].

The original variational formulation of the hybrid stress model[3, 4] was limited to static equilibrium problems for which consistent nodal forces due to body force and initial strain can be derived. It is apparent that in the hybrid stress model, it is no longer possible to adopt the procedure in the conventional assumed displacement approach to account for in a consistent manner, the inertia force in the dynamic problems or the geometric stiffness in the buckling or large deflection problems. Tabarrok[11] has applied the hybrid stress model successfully to the free vibration problem by employing the Toupin principle. However, his formulation reduced to the determination of the singular condition of a dynamic stiffness matrix which is a function of the vibration frequency. Thus, the determination of the natural modes and frequencies cannot be treated as a standard eigenvalue problem and his formulation is not adaptable to the transient response analysis. Recently, Atluri[12] presented a consistent variational formulation of hybrid stress finite element model for the solution of linear transient response. His formulation is based on the use

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of convolution integrals in time, and the resulting equations, again, do not contain mass and stiffness matrices. Other authors have analyzed the vibration problem by using the stiffness matrix of the hybrid model together with a mass matrix obtained by using independently assumed displacement functions or by simple lumping[13]. This scheme can be extended to transient analyses. Such an approach, however, has been criticized for its inconsistency. A similar situation exists in the case of large displacement or buckling problems[14], except in the solutions by Pirotin[15] and Allman[16] who have used a form of Hellinger–Reissner Principle to construct the geometric stiffness matrices. The assumed displacement hybrid model initiated by Tong[17] has also been extended to dynamics or buckling problems[18]. The problem of constructing appropriate mass matrices for the equilibrium model[19] has been discussed by Geradin[20].

In this paper, we shall illustrate a unifying approach of treating the dynamics problems including initial stresses by the hybrid models, point out the various options and spread a word of caution to some of the options. Two by-products of the present study are: (a) the establishment of a rational way to assess the element interchangability in the finite element method; (b) the suggestion of an alternative way to obtain the loading vector for the hybrid stress model.

2. FORMULATION IN THE FORM OF THE REISSNER PRINCIPLE

The functional for a modified Reissner Principle[21,22] can be written in the form

$$\pi_R = \sum_n \int_{t_1}^{t_2} \left\{ \int_{V_n} [\sigma_{ij} e_{ij} - \tfrac{1}{2} S_{ijkl} \sigma_{ij} \sigma_{kl} - F_i u_i + \tfrac{1}{2} \sigma^0_{kj} u_{i,k} u_{i,j} - \tfrac{1}{2} \rho \dot{u}_i^2] \, \mathrm{d}V + \int_{\partial V_n} T_i (\tilde{u}_i - u_i) \, \mathrm{d}S - \int_{S_{\sigma_n}} \bar{T}_i \tilde{u}_i \, \mathrm{d}S \right\} \mathrm{d}t \tag{1}$$

where

σ_{ij} = stress tensor
σ^0_{ij} = initial stress tensor
S_{ijkl} = elastic compliance tensor
u_i = displacement within a subregion
e_{ij} = strain tensor
$\tilde{u}_i$ = displacement on the boundary of a subregion
T_i = force on ∂V_n
$\bar{T}_i$ = boundary traction
V_n = volume of the nth subregion
∂V_n = boundary of the nth subregion
S_{σ_n} = surface of the nth subregion over which traction is prescribed
$(\dot{\ })$ = time derivative, $(\mathrm{d}/\mathrm{d}t)$
F_i = body force

subjecting to the constraints that $\tilde{u}_i = \bar{u}_i$ over the boundaries where the displacement is prescribed as $\bar{u}_i$ and that u_i satisfy prescribed conditions at the limits $t = t_1$ and $t = t_2$, all the quantities σ_{ij}, T_i, u_i and $\tilde{u}_i$ can vary independently. It is understood that the symmetric components of the stress tensor are considered as independent quantities. The validity of

Reissner's function is, of course, based on the existence of the complementary energy density and the corresponding constitutive relations. The first variation of π_R is

$$\delta\pi_R = \sum_n \int_{t1}^{t2} \left\{ \int_{V_n} \delta u_i [\rho \ddot{u}_i - \sigma_{ij,j} - F_i - (\sigma_{kj}^0 u_{i,j})_{,k}] \, \mathrm{d}V \right.$$
$$+ \int_{\partial V_n} \delta u_i (\sigma_{ij} \nu_j + \sigma_{kj}^0 \nu_k u_{i,j} - T_i) \, \mathrm{d}S \tag{2}$$
$$+ \int_{V_n} \delta\sigma_{ij} [\tfrac{1}{2}(u_{i,j} + u_{j,i}) - S_{ijkl}\sigma_{kl}] \, \mathrm{d}V + \int_{\partial V_n} \delta T_i (\tilde{u}_i - u_i) \, \mathrm{d}S$$
$$\left. + \int_{\partial V_n} \delta\tilde{u}_i T_i \, \mathrm{d}S + \int_{S_{\sigma_n}} \delta\tilde{u}_i \bar{T}_i \, \mathrm{d}S \right\} \mathrm{d}t.$$

From $\delta\pi_R = 0$, we obtain the following Euler equations

$$\rho \ddot{u}_i = \sigma_{ij,j} + (\sigma_{kj}^0 u_{i,j})_{,k} + F_i \tag{3}$$

$$\tfrac{1}{2}(u_{i,j} + u_{j,i}) = S_{ijkl}\sigma_{kl} \tag{4}$$

in V_n,

$$T_i = \sigma_{ij}\nu_j + \sigma_{kl}^0 \nu_k u_{i,j} \tag{5}$$

$$u_i = \tilde{u}_i \tag{6}$$

on ∂V_n,

$$(T_i)_{\mathrm{I}} = (T_i)_{\mathrm{II}} \tag{7}$$

over the common boundaries of any two adjacent elements I and II and

$$T_i = \bar{T}_i \tag{8}$$

on S_{σ_n},

3. FINITE-ELEMENT FORMULATION

By recognizing that σ_{ij}, T_i, u_i within each element and $\tilde{u}_i$ along all of the interelement boundaries and on ∂V_σ can be arbitrarily varied, in the most general form, we can assume σ_{ij}, T_i, u_i and $\tilde{u}_i$ independently in terms of unknown constants and to construct the corresponding finite-element equations. These will lead to many unknown variables, which may, or may not be useful in practice from the point of view of accuracy and efficiency. To reduce the unknown variables, we can require some of the relations in equations (3–8) be satisfied indentically. There are many options. We shall restrict our discussion to two cases which we feel may be of practical interest. These are the hybrid displacement model and a hybrid mixed model.

(a) *Hybrid displacement model*

By requiring equation (4) to be satisfied exactly with σ_{ij} expressed in terms of u_i, equation (1) becomes

$$\pi_D = \sum_n \int_{t_1}^{t_2} \left\{ \int_{V_n} [\tfrac{1}{2}C_{ijkl} e_{ij} e_{kl} + \tfrac{1}{2}\sigma_{ij}^0 u_{k,i} u_{k,j} - F_i u_i - \tfrac{1}{2}\rho \dot{u}_i^2] \, \mathrm{d}V \right.$$
$$\left. + \int_{\partial V_n} T_i (\tilde{u}_i - u_i) \, \mathrm{d}S - \int_{S_{\sigma_n}} \bar{T}_i \tilde{u}_i \, \mathrm{d}S \right\} \mathrm{d}t \tag{9}$$

where C_{ijkl} is the elastic coefficient and e_{ij} is in terms of u_i. In the static equilibrium problem this reduced to the finite element model first suggested by Tong[16]. In constructing the finite-element equations, we assume, for the nth element

$$\begin{aligned} \boldsymbol{T} &= \boldsymbol{R\alpha} \\ \boldsymbol{u} &= \boldsymbol{U}\gamma \\ \boldsymbol{e} &= \boldsymbol{D}\gamma \\ \tilde{\boldsymbol{u}} &= \boldsymbol{Lq} \end{aligned} \tag{10}$$

where $\boldsymbol{R}$, $\boldsymbol{U}$, $\boldsymbol{D}$ and $\boldsymbol{L}$are functions of x, y, z in which $\boldsymbol{R}$ and $\boldsymbol{L}$are defined only on the element boundaries. $\boldsymbol{T}$, $\boldsymbol{u}$, and $\boldsymbol{e}$ are, respectively, the vectors for the boundary traction, the displacement and the strain, $\boldsymbol{D}$ is derived from $\boldsymbol{U}$ by the use of strain displacement relations, and $\tilde{\boldsymbol{u}}$ is the vector for the interelement boundary displacements. The form of $\boldsymbol{u}$ and $\boldsymbol{T}$ can be different for different elements. $\tilde{\boldsymbol{u}}$ are the same for two adjacent elements at the common boundaries. A substitution of equation (10) into equation (9) yields

$$\pi_D = \sum_n \int_{t_1}^{t_2} [\tfrac{1}{2}\gamma^T \boldsymbol{A}\gamma + \tfrac{1}{2}\gamma^T \boldsymbol{B}\gamma + \boldsymbol{\alpha}^T(\boldsymbol{G}_1\boldsymbol{q} - \boldsymbol{G}_2\gamma) - \tfrac{1}{2}\dot{\gamma}^T \boldsymbol{m}_1\dot{\gamma} - \gamma^T \boldsymbol{Q}_2 - \boldsymbol{q}^T\boldsymbol{Q}_1]\,\mathrm{d}t. \tag{11}$$

In which

$$\begin{aligned} \boldsymbol{A} &= \int_{V_n} \boldsymbol{D}^T\boldsymbol{C}\boldsymbol{D}\,\mathrm{d}V \\ \gamma^T\boldsymbol{B}\gamma &= \int_{V_n} \sigma^0_{kj}u_{i,k}u_{i,j}\,\mathrm{d}V \\ \boldsymbol{G}_1 &= \int_{\partial V_n} \boldsymbol{R}^T\boldsymbol{L}\,\mathrm{d}S,\ \boldsymbol{G}_2 = \int_{\partial V_n} \boldsymbol{R}^T\boldsymbol{U}\,\mathrm{d}S \\ \boldsymbol{m}_1 &= \int_{V_n} \rho\boldsymbol{U}^T\boldsymbol{U}\,\mathrm{d}V \\ \boldsymbol{Q}_1 &= \int_{S_{\sigma_n}} \boldsymbol{L}^T\bar{\boldsymbol{T}}\,\mathrm{d}S \\ \boldsymbol{Q}_2 &= \int_{V_n} \boldsymbol{U}^T\boldsymbol{F}\,\mathrm{d}V \end{aligned} \tag{12}$$

where $\boldsymbol{C}$ is an elastic constant matrix.

By varying π_D with respect to γ and $\boldsymbol{\alpha}$ we obtained

$$(\boldsymbol{A} + \boldsymbol{B})\gamma - \boldsymbol{G}_2\boldsymbol{\alpha} - \boldsymbol{Q}_2 + \boldsymbol{m}_1\ddot{\gamma} = 0 \tag{13}$$

$$\boldsymbol{G}_1\boldsymbol{q} - \boldsymbol{G}_2\gamma = 0 \tag{14}$$

for the nth element. In general, unlike the static equilibrium problems $\boldsymbol{\alpha}$ and γ cannot be eliminated in terms of $\boldsymbol{q}$ by using equations (13) and (14), so that the final equations are in terms of $\boldsymbol{q}$ alone. However, since we are completely free in approximating the boundary displacements and tractions we may choose $\boldsymbol{u}$ and $\boldsymbol{T}$ in a special way such that $\boldsymbol{G}_2$ is a nonsingular square matrix, from equation (14) we have

$$\gamma = G_2^{-1} G_1 q. \tag{15}$$

A substitution of equation (15) into (11) yields

$$\pi_D = \sum_n \int_{t_1}^{t_2} (\tfrac{1}{2} q^T k q - q^T Q - \tfrac{1}{2} \dot{q}^T m \dot{q})\, dt \tag{16}$$

in which

$$\begin{aligned} k &= k_1 + k_2,\; k_1 = (G_1 G_2^{-1})^T A (G_1 G_2^{-1}) \\ k_2 &= (G_1 G_2^{-1})^T B (G_1 G_2^{-1}) \\ m &= (G_1 G_2^{-1})^T m_1 G_2^{-1} G_1 \\ Q &= (G_1 G_2^{-1})^T Q_2 + Q_1. \end{aligned} \tag{17}$$

The matrices $\boldsymbol{k}$, $\boldsymbol{m}$, and $\boldsymbol{Q}$ are the stiffness matrix, the mass matrix and the load vector for the element. $\boldsymbol{k}_1$ is the element stiffness matrix under no applied initial stresses. $\boldsymbol{k}_2$ is the so-called element geometric stiffness matrix.

One of the simplest ways of choosing $\boldsymbol{u}$ and $\boldsymbol{T}$ so that $\boldsymbol{G}_2$ is a square nonsingular matrix, is to have

$$R = U|_{\text{at } \partial V_n} \tag{18}$$

in which $\boldsymbol{u}_{\text{at } \partial V_n}$ is not identically zero unless $\gamma \equiv 0$. The mathematical implication for this choice is to require a least mean square fit of $\boldsymbol{u}$ and $\tilde{\boldsymbol{u}}$ over the element boundaries.

(b) *Hybrid mixed model*

We separate T_i and σ_{ij} each into two parts, i.e.

$$\begin{aligned} T_i &= T_i^H + T_i^F \\ \sigma_{ij} &= \sigma_{ij}^H + \sigma_{ij}^F \end{aligned} \tag{19}$$

where

$$\begin{aligned} \sigma_{ij,j}^H &= 0 \\ \sigma_{ij,j}^F &\not\equiv 0 \end{aligned} \tag{20}$$

and specifically we require

$$T_i^H = \sigma_{ij}^H \nu_j \tag{21}$$

at the element boundaries. By so doing, equation (1) becomes

$$\begin{aligned} \pi_H = \sum_n \int_{t_1}^{t_2} \Bigg\{ & \int_{V_n} [\sigma_{ij}^F e_{ij} - \tfrac{1}{2} S_{ijkl} \sigma_{ij} \sigma_{kl} - F_i u_i + \tfrac{1}{2} \sigma_{ij}^0 u_{k,l} u_{k,j} - \tfrac{1}{2} \rho \dot{u}_i^2]\, dV \\ & + \int_{\partial V_n} T_i^H \tilde{u}_i \, dS + \int_{\partial V_n} T_i^F (\tilde{u}_i - u_i)\, dS \\ & - \int_{S_{\sigma_n}} \bar{T}_i \tilde{u}_i \, dS \Bigg\}\, dt. \end{aligned} \tag{22}$$

We shall assume

$$\begin{aligned}
\boldsymbol{\sigma}^H &= \boldsymbol{P}_H \boldsymbol{\beta}_1 \\
\boldsymbol{\sigma}^F &= \boldsymbol{P}_F \boldsymbol{\beta}_2 \\
\boldsymbol{T}^H &= \boldsymbol{R}_H \boldsymbol{\beta}_1 \\
\boldsymbol{T}^F &= \boldsymbol{R}\boldsymbol{\alpha} \\
\boldsymbol{u} &= \boldsymbol{U}\boldsymbol{\gamma} \\
\boldsymbol{e} &= \boldsymbol{D}\boldsymbol{\gamma} \\
\tilde{\boldsymbol{u}} &= \boldsymbol{L}\boldsymbol{q}.
\end{aligned} \tag{23}$$

$\boldsymbol{P}_H$ and $\boldsymbol{R}_H$ are chosen so that equations (20) and (21) are satisfied. A substitution of equation (23) into (22) yields

$$\begin{aligned}
\pi_H = \sum_n \int_{t_1}^{t_2} \Big\{ & \boldsymbol{\beta}_2^T \boldsymbol{S}_2 \boldsymbol{\gamma} - (\tfrac{1}{2}\boldsymbol{\beta}_1^T \boldsymbol{H}_1 \boldsymbol{\beta}_1 + \boldsymbol{\beta}_1^T \boldsymbol{H}_{12} \boldsymbol{\beta}_2 + \tfrac{1}{2}\boldsymbol{\beta}_2^T \boldsymbol{H}_2 \boldsymbol{\beta}_2) - \boldsymbol{\gamma}^T \boldsymbol{Q}_2 \\
& + \tfrac{1}{2}\boldsymbol{\gamma}^T \boldsymbol{A}\boldsymbol{\gamma} - \tfrac{1}{2}\dot{\boldsymbol{\gamma}}^T \boldsymbol{m}_1 \dot{\boldsymbol{\gamma}} + \boldsymbol{\beta}_1^T \boldsymbol{G}\boldsymbol{q} \\
& + \boldsymbol{\alpha}^T(\boldsymbol{G}_1 \boldsymbol{q} - \boldsymbol{G}_2 \boldsymbol{\gamma}) - \boldsymbol{q}^T \boldsymbol{Q}_1 \Big\} \, \mathrm{d}t
\end{aligned} \tag{24}$$

in which

$$\begin{aligned}
\boldsymbol{S}_2 &= \int_{V_n} \boldsymbol{P}_F^T \boldsymbol{D} \, \mathrm{d}V \\
\boldsymbol{H}_1 &= \int_{V_n} \boldsymbol{P}_H^T \boldsymbol{S} \boldsymbol{P}_H \, \mathrm{d}V, \ \boldsymbol{H}_{12} = \int_{V_n} \boldsymbol{P}_H^T \boldsymbol{S} \boldsymbol{P}_F \, \mathrm{d}V, \ \boldsymbol{H}_2 = \int_{V} \boldsymbol{P}_F^T \boldsymbol{S} \boldsymbol{P}_F \, \mathrm{d}V \\
\boldsymbol{G} &= \int_{\partial V_n} \boldsymbol{R}_H^T \boldsymbol{L} \, \mathrm{d}S
\end{aligned} \tag{25}$$

and the rest of the matrices are in the similar form as equation (12).

The variation of π_H with respect to $\boldsymbol{\alpha}$, $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$ and $\boldsymbol{\gamma}$ yields

$$\begin{aligned}
\boldsymbol{G}_1 \boldsymbol{q} - \boldsymbol{G}_2 \boldsymbol{\gamma} &= 0 \\
\boldsymbol{H}_1 \boldsymbol{\beta}_1 + \boldsymbol{H}_{12} \boldsymbol{\beta}_2 &= \boldsymbol{G}\boldsymbol{q} \\
\boldsymbol{H}_2 \boldsymbol{\beta}_2 + \boldsymbol{H}_{12}^T \boldsymbol{\beta}_1 - \boldsymbol{S}_2 \boldsymbol{\gamma} &= 0 \\
\boldsymbol{m}_1 \ddot{\boldsymbol{\gamma}} + \boldsymbol{S}_2^T \boldsymbol{\beta}_2 + \boldsymbol{A}\boldsymbol{\gamma} &= \boldsymbol{G}_2^T \boldsymbol{\alpha} + \boldsymbol{Q}_2 .
\end{aligned}$$

If we also choose $\boldsymbol{R}$ and $\boldsymbol{U}$ in equation (23) such that $\boldsymbol{G}_2$ is a square nonsingular matrix, we have

$$\begin{aligned}
\boldsymbol{\gamma} &= \boldsymbol{G}_2^{-1} \boldsymbol{G}_1 \boldsymbol{q} \\
\boldsymbol{\beta}_1 &= \boldsymbol{H}_1^{-1} \boldsymbol{G}\boldsymbol{q} - \boldsymbol{H}_1^{-1} \boldsymbol{H}_{12} \boldsymbol{\beta}_2 \\
\boldsymbol{\beta}_2 &= (\boldsymbol{H}_2 - \boldsymbol{H}_{12}^T \boldsymbol{H}_1^{-1} \boldsymbol{H}_{12})^{-1} (\boldsymbol{S}_2 \boldsymbol{G}_2^{-1} \boldsymbol{G}_1 - \boldsymbol{H}_{12} \boldsymbol{H}_1^{-1} \boldsymbol{G}) \boldsymbol{q}.
\end{aligned} \tag{27}$$

A substitution of equation (27) into (24) reduces π_H in the same form as equation (16) in which

$$
\begin{aligned}
\boldsymbol{k} &= \boldsymbol{k}_1 + \boldsymbol{k}_2 \\
\boldsymbol{k}_1 &= \boldsymbol{G}^T\boldsymbol{H}_1^{-1}\boldsymbol{G} + \boldsymbol{B}^T(\boldsymbol{H}_2 - \boldsymbol{H}_{12}^T\boldsymbol{H}_1^{-1}\boldsymbol{H}_{12})^{-1}\boldsymbol{B} \\
\boldsymbol{B} &= \boldsymbol{S}_2\boldsymbol{G}_2^{-1}\boldsymbol{G}_1 - \boldsymbol{H}_{12}\boldsymbol{H}_1^{-1}\boldsymbol{G} \\
\boldsymbol{k}_2 &= (\boldsymbol{G}_2^{-1}\boldsymbol{G}_1)^T\boldsymbol{A}(\boldsymbol{G}_2^{-1}\boldsymbol{G}_1) \\
\boldsymbol{m} &= (\boldsymbol{G}_2^{-1}\boldsymbol{G}_1)^T\boldsymbol{m}_1(\boldsymbol{G}_2^{-1}\boldsymbol{G}_1) \\
\boldsymbol{Q} &= \boldsymbol{Q}_1 + (\boldsymbol{G}_2^{-1}\boldsymbol{G}_1)^T\boldsymbol{Q}_2 .
\end{aligned}
\tag{28}
$$

The present mass matrix $\boldsymbol{m}$ and geometric stiffness matrix $\boldsymbol{k}_2$ reduce to those of the regular displacement model if $u_i = \tilde{u}_i$.

It should be noted that the load vector is different from that derived in[3] and[4]. This is because the inhomogeneous equation is, in general, not satisfied exactly. This can be viewed as an alternative way to derive the loading vector for the hybrid stress model. It should also be noted that the first term in $\boldsymbol{k}_1$ is the original hybrid stress element stiffness matrix derived by Pian, the second term is due to assuming non-equilibrium stresses within the elements and $\boldsymbol{k}_2$ is of course due to initial stresses.

The expression of $\boldsymbol{k}$ in equation (28) is quite complicated. In practice, we may, in equation (23), choose $\boldsymbol{\sigma}^F$ to be zero or to be in the form such that $\sigma^F_{ij,j} = F_i$. In the former case $\boldsymbol{\beta}_2 = 0$ and in the latter case $\boldsymbol{\beta}_2$ is prescribed. In both cases $\boldsymbol{\beta}_2$ is no longer a variable, hence the third equation of (26) and the last equation of (27) no longer appear. The stiffness matrix $\boldsymbol{k}$ reduces to that of the original hybrid stress model when initial stresses are zero. Apparently this is an approximation. Physically, this is equivalent to a model in which a lattice structure along the element boundaries are carrying all of the initial stresses and all of the masses. Another possible choice is simply to put $\boldsymbol{\sigma}^H$ to be zero. In that case, $\boldsymbol{\beta}_1$ is no longer a variable and $\boldsymbol{k}_1$ becomes

$$\boldsymbol{k}_1 = (\boldsymbol{S}_2\boldsymbol{G}_2^{-1}\boldsymbol{G}_1)\boldsymbol{H}_2^{-1}(\boldsymbol{S}_2\boldsymbol{G}_2^{-1}\boldsymbol{G}_1). \tag{29}$$

This is the same as the use of the original form of equation (1) in deriving $\boldsymbol{k}$.

4. FUNCTIONAL FOR PLATE BENDING

The corresponding function of equation (1) for a plate will be

$$
\begin{aligned}
\pi_R = \sum_n \int_{t_1}^{t_2} \Bigg\{ \iint_{A_n} & (M_{\alpha\beta} w_{,\alpha\beta} - \tfrac{1}{2} S_{\alpha\beta\lambda\theta} M_{\alpha\beta} M_{\lambda\theta} + \tfrac{1}{2} N^0_{\alpha\beta} w_{,\alpha} w_{,\beta} \\
& - pw - \tfrac{1}{2}\rho\dot{w}^2)\, \mathrm{d}A + \int_{\partial A_n} [m_\alpha(\tilde{w}_{,\alpha} - w_{,\alpha}) - Q(\tilde{w} - w)]\, \mathrm{d}s \\
& - \int_{S_{\sigma_n}} (\overline{m}_\alpha \tilde{w}_{,\alpha} - \overline{Q}\tilde{w})\, \mathrm{d}s \Bigg\} \mathrm{d}t
\end{aligned}
\tag{30}
$$

where $M_{\alpha\beta}$, w and $N^0_{\alpha\beta}$ are the moments, the displacement and the initial midplane stress resultants, m_α and Q are the boundary moment and shear of the elements and $\tilde{w}$ is the interelement boundary displacement. In the most general form, all the quantities above are the independent field variables.

By requiring the moment displacement relations

$$w_{,\alpha\beta} = S_{\alpha\beta\lambda\theta} M_{\lambda\theta}$$

to be satisfied exactly. Equation (30) reduces to the functional for the hybrid displacement model; namely

$$\pi_D = \sum_n \int_{t_1}^{t_2} \left\{ \int_{A_n} [\tfrac{1}{2} C_{\alpha\beta\lambda\theta} w_{,\alpha\beta} w_{,\lambda\theta} + \tfrac{1}{2} N^0_{\alpha\beta} w_{,\alpha} w_{,\beta} - pw - \tfrac{1}{2}\rho \dot{w}^2] \, \mathrm{d}A \right.$$
$$+ \int_{\partial A_n} [m_\alpha(\tilde{w}_{,\alpha} - w_{,\alpha}) - Q(\tilde{w} - w)] \, \mathrm{d}s$$
$$\left. - \int_{S_{\sigma_n}} (\bar{m}_\alpha \tilde{w}_{,\alpha} - \bar{Q}\tilde{w}) \, \mathrm{d}s \right\} \mathrm{d}t. \tag{31}$$

Another version of the finite element model can be constructed similar to that of Section 3(b). We separate

$$\begin{aligned} M_{\alpha\beta} &= M^H_{\alpha\beta} + M^F_{\alpha\beta} \\ Q &= Q^H + Q^F \\ m_\alpha &= m^H_\alpha + m^F_\alpha \end{aligned} \tag{32}$$

where

$$\begin{aligned} M^H_{\alpha\beta,\alpha\beta} &= 0 \\ M^F_{\alpha\beta,\alpha\beta} &\neq 0. \end{aligned} \tag{33}$$

We shall require that

$$\begin{aligned} m^H_\alpha &= M^H_{\alpha\beta} \nu_\beta \\ Q^H &= M^H_{\alpha\beta,\beta} \nu_\alpha . \end{aligned} \tag{34}$$

A substitution of equations (32–34) into (30) yields

$$\pi_H = \sum_n \int_{t_1}^{t_2} \left\{ \int_{A_n} [M^F_{\alpha\beta} w_{,\alpha\beta} - \tfrac{1}{2} S_{\alpha\beta\lambda\theta} M_{\alpha\beta} M_{\lambda\theta} + \tfrac{1}{2} N^0_{\alpha\beta} w_{,\alpha} w_{,\beta} - pw - \tfrac{1}{2}\rho\dot{w}^2] \, \mathrm{d}A \right.$$
$$+ \int_{\partial A_n} [m^H_\alpha \tilde{w}_{,\alpha} - Q^H \tilde{w} + m^F_\alpha(\tilde{w}_{,\alpha} - w_{,\alpha}) - Q^F(\tilde{w} - w)] \, \mathrm{d}s$$
$$\left. + \int_{S_{\sigma_n}} (\bar{m}_\alpha \tilde{w}_{,\alpha} - \bar{Q}\tilde{w}) \, \mathrm{d}s \right\} \mathrm{d}t. \tag{35}$$

The construction of the finite-element equations can be performed in the routine manner. However, if we want to be able to eliminate all of the unknown constants for $M_{\alpha\beta}$, w, Q and m_α in terms of those of $\tilde{w}$, we must choose the number of unknowns for Q and m_α (or Q^F and m^F in the case of equation 35) to be the same as that for w and have

$$\int_{\partial A_n} (-Qw + m_\alpha w_{,\alpha}) \, \mathrm{d}s \neq 0$$

unless w and $w_{,\alpha}$ or Q and m_α (or Q^F and M_α^F) are identically zero. In other words, suppose we assume that on ∂A_n

$$Q = \boldsymbol{R}_1\boldsymbol{\alpha}$$
$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \boldsymbol{R}_2\boldsymbol{\alpha} \tag{36}$$

and

$$w = \boldsymbol{U}_1\boldsymbol{\gamma}$$
$$\begin{pmatrix} w_{,1} \\ w_{,2} \end{pmatrix} = \boldsymbol{U}_2\boldsymbol{\gamma} \tag{37}$$

where $\boldsymbol{R}$'s and $\boldsymbol{U}$'s are functions of x and y on ∂A_n, then

$$\int_{\partial A_n} (-Qw + m_\alpha w_{,\alpha})\,\mathrm{d}s = \boldsymbol{\alpha}^T(\boldsymbol{G}_3 + \boldsymbol{G}_4)\boldsymbol{\gamma} = \boldsymbol{\alpha}^T\boldsymbol{G}_2\boldsymbol{\gamma}$$

where

$$\boldsymbol{G}_3 = \int_{\partial A_n} \boldsymbol{R}_1^T\boldsymbol{U}_1\,\mathrm{d}s$$
$$\boldsymbol{G}_4 = \int_{\partial A_n} \boldsymbol{R}_2^T\boldsymbol{U}_2\,\mathrm{d}s. \tag{38}$$

The vectors $\boldsymbol{R}_1$, $\boldsymbol{R}_2$, $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$ must be chosen so that $\boldsymbol{G}$ is a square nonsingular matrix. One may choose $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ similar to that of equation (18) with a slight modification, i.e.

$$\boldsymbol{R}_1 = \boldsymbol{U}_1$$
$$\boldsymbol{R}_2 = \varepsilon^2\boldsymbol{U}_2 \tag{39}$$

where ε is the linear dimension of the element. The additional factor for $\boldsymbol{R}_2$ is to make $\boldsymbol{G}_3$ and $\boldsymbol{G}_4$ have the same physical dimensions to avoid large and small numbers in $\boldsymbol{G}_3$ and $\boldsymbol{G}_4$.

The extension of the present formulation to shell is trivial.

5. EXAMPLES

We shall consider some simple examples of a beam with homogeneous boundary conditions. In this case, if we require that the element be kinematically stable, and want to be able to eliminate all the interior degrees of freedom of each element, the hybrid displacement model will reduce to the regular compatible model. (This is, in general, not true in the two-dimensional problems.) Therefore we shall only illustrate the example by the hybrid stress model which, in this special case, is reduced to the equilibrium model of Fraeijs de Veubeke[19]. The functional for the beam is

$$\pi_H = \sum_n \int_{t_1}^{t_2} \left\{ \int_{x_n}^{x_{n+1}} \left[M^F \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} - \frac{1}{2}\frac{(M^H + M^F)^2}{EI} - \frac{N}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 - \tfrac{1}{2}\rho \dot{w}^2 \right] \mathrm{d}x \right.$$
$$\left. + \left[-\frac{\mathrm{d}M^H}{\mathrm{d}x}\tilde{w} + M^H\tilde{\theta} - Q^F(\tilde{w} - w) + m^F\left(\tilde{\theta} - \frac{\mathrm{d}w}{\mathrm{d}x}\right)\right]_{x_n}^{x_{n+1}} \right\} \mathrm{d}t \tag{40}$$

where M^H and M^F are the homogeneous and particular part of the assumed moment in the element, and Q^F and m^F are the boundary shear and moment corresponding to $\boldsymbol{\alpha}$ in equation (23). We assume

$$M^H = \beta_1 + \frac{x}{\varepsilon}\beta_2 \qquad M^F = 0 \tag{41}$$

and

$$\tilde{\boldsymbol{u}}^T = \boldsymbol{q}^T = (w_{i-1}\ \varepsilon\theta_{i-1}\ w_i\ \varepsilon\theta_i)^T. \tag{42}$$

Where ε is the length of an element and w, θ are the nodal displacement and slope respectively.

Using equations (41) and (42), and substituting into (28) we obtain

$$\boldsymbol{k}_1 = \boldsymbol{G}^T\boldsymbol{H}_1^{-1}\boldsymbol{G} = \frac{EI}{\varepsilon^3}\begin{pmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{pmatrix} \tag{43}$$

which is identical to the stiffness matrix obtained from assumed displacement method.

To find the mass matrix $\boldsymbol{m}$ and the geometric stiffness matrix $\boldsymbol{k}_2$, the following seven cases are tried:

Case 1. Constant $\boldsymbol{u}$, and equal shear force Q^F.

Case 2. Two constant $\boldsymbol{u}$ segments, and two different Q^F's. (The case maintains the same simplicity as in Case 1, that $\boldsymbol{u}$ is piecewise constant but has different values at the ends of each element.)

Case 3. Linear $\boldsymbol{u}$, and two different Q^F's.

Case 4. Linear $\boldsymbol{u}$, and equal Q^F and m^F.

Case 5. Bilinear $\boldsymbol{u}$, and equal Q^F and two different m^F.

Case 6. Quadratic $\boldsymbol{u}$, and two different Q^F's and one m^F.

Case 7. Cubic $\boldsymbol{u}$, and two Q^F and two m^F.

The assumed $\boldsymbol{u}$ and $\boldsymbol{T}^F$ in these cases are illustrated in Fig. 1†. From these assumed $\boldsymbol{u}$, Q^F and m^F, the matrices $\boldsymbol{m}$ and $\boldsymbol{k}_2$ can be calculated and the results are summarized in Table 1.

For a simply supported beam, i.e. $w = 0$ at both ends, the associated finite element eigenvalue problems for vibration and buckling can be solved exactly if we assume a uniform mesh[23]. To illustrate the solution procedure, case 3 is solved as follows:

(a) *Vibration*

The governing equations are

$$\begin{gathered} -12w_{j-1} + 24w_j - 12w_{j+1} + 6\varepsilon(\theta_{j+1} - \theta_{j-1}) = \lambda\varepsilon^4(\tfrac{1}{6}w_{j-1} + \tfrac{2}{3}w_j + \tfrac{1}{6}w_{j+1}) \\ 6\varepsilon(w_{j-1} - w_{j+1}) + 2\varepsilon^2(\theta_{j-1} + 4\theta_j + \theta_{j+1}) = 0 \end{gathered} \tag{44}$$

where

$$\lambda = \frac{\rho A}{EI}\omega^2$$

and ω, ρ, A are the frequency, density and cross-sectional area of the beam respectively.

† The boundary forces (α's) are chosen to make $\boldsymbol{G}_2$ nonsingular.

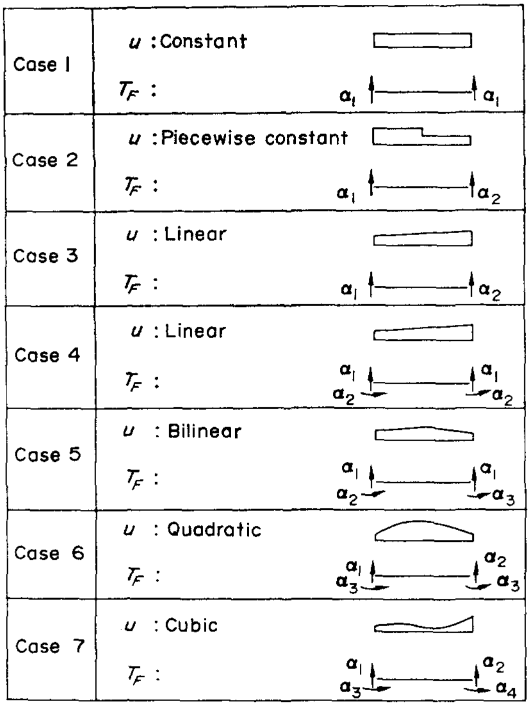

Fig. 1. Displacement and boundary force assumptions for the beam example.

To solve equation (44) we try as a solution

$$w_j = a\sin(j\alpha\varepsilon)$$
$$\theta_j = b\cos(j\alpha\varepsilon) \tag{45}$$

with $\sin(n\alpha\varepsilon) = 0$ or $\alpha = \alpha_r = \dfrac{r\pi}{n\varepsilon} = \dfrac{r\pi}{L}$, $r = 1, 2 \ldots n-1$. Substituting equation (45) into (44) and seeking a nontrivial solution to the resulting homogeneous equations in a and b, we find λ_r^* must satisfy the following equation

$$[\cos(\alpha_r\varepsilon) + 2]^2\lambda_r^*\varepsilon^4 - 36[1 - \cos(\alpha_r\varepsilon)]^2 = 0.$$

If $\alpha_r\varepsilon$ is small, an expansion of the trigonometric function yields

$$E_{rV} = \frac{\lambda_r^* - \lambda_{rV}}{\lambda_{rV}} = \tfrac{1}{6}\varepsilon^2\lambda_{rV}^{1/2} + 0(\varepsilon^4\lambda_{rV}) \tag{46}$$

where E_{rV} is the error in the rth eigenvalue of the beam vibration problem and λ_{rV} is the exact solution $\lambda_{rV} = \left(\dfrac{r\pi}{n\varepsilon}\right)^4$.

Table 1. Results for the beam example

	$\boldsymbol{m}$: $\rho A \varepsilon$	$\boldsymbol{k}_2$: EI/ε^3	Error†
Case 1	$\begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$-N\,[0]$	$E_{rv} = -\frac{7}{12}\varepsilon^2\lambda_{rv}^{1/2}$ $E_{rB} = \infty$
Case 2	$\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$-N\,[0]$	$E_{rv} = \frac{1}{80}\varepsilon^4\lambda_{rv}$ $E_{rB} = \infty$
Case 3	$\begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{6} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$-N\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$E_{rv} = \frac{1}{6}\varepsilon^2\lambda_{rv}^{1/2}$ $E_{rB} = \frac{1}{12}\varepsilon^2\lambda_{rB}$
Case 4	$\begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{48} & 0 & \frac{1}{48} \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{48} & 0 & \frac{1}{48} \end{pmatrix}$	$-N\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} \end{pmatrix}$	$E_{rv} = \frac{1}{6}\varepsilon^2\lambda_{rv}^{1/2}$ $E_{rB} = \frac{1}{3}\varepsilon^2\lambda_{rB}$
Case 5	$\begin{pmatrix} \frac{1}{4} & \frac{1}{16} & \frac{1}{4} & -\frac{1}{16} \\ \frac{1}{16} & \frac{1}{24} & \frac{1}{16} & 0 \\ \frac{1}{4} & \frac{1}{16} & \frac{1}{4} & -\frac{1}{16} \\ -\frac{1}{16} & 0 & -\frac{1}{16} & \frac{1}{24} \end{pmatrix}$	$-N\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$	$E_{rv} = -\frac{1}{12}\varepsilon^2\lambda_{rv}^{1/2}$ $E_{rB} = -\frac{1}{12}\varepsilon^2\lambda_{rB}$
Case 6	$\begin{pmatrix} \frac{1}{3} & \frac{1}{24} & \frac{1}{6} & -\frac{1}{24} \\ \frac{1}{24} & \frac{1}{120} & \frac{1}{24} & -\frac{1}{120} \\ \frac{1}{6} & \frac{1}{24} & \frac{1}{5} & -\frac{1}{24} \\ -\frac{1}{24} & -\frac{1}{120} & -\frac{1}{24} & \frac{1}{120} \end{pmatrix}$	$-N\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & \frac{1}{12} & 0 & -\frac{1}{12} \\ -1 & 0 & 1 & 0 \\ 0 & -\frac{1}{12} & 0 & \frac{1}{12} \end{pmatrix}$	$E_{rv} = \frac{1}{45}\varepsilon^4\lambda_{rv}$ $E_{rB} = \frac{1}{360}\varepsilon^4\lambda_{rB}^2$
Case 7	$\begin{pmatrix} 156 & 22 & 54 & -13 \\ 22 & 4 & 13 & -3 \\ 54 & 13 & 156 & -22 \\ -13 & -3 & -22 & 4 \end{pmatrix}$	$-N\begin{pmatrix} \frac{1}{6} & \frac{1}{10} & -\frac{6}{5} & \frac{1}{10} \\ \frac{1}{10} & \frac{2}{15} & -\frac{1}{10} & \frac{1}{30} \\ -\frac{6}{5} & \frac{1}{10} & \frac{1}{6} & -\frac{1}{10} \\ \frac{1}{10} & \frac{1}{30} & \frac{1}{10} & \frac{2}{15} \end{pmatrix}$	$E_{rv} = \frac{1}{720}\varepsilon^4\lambda_{rv}$ $E_{rB} = \frac{1}{1200}\varepsilon^4\lambda_{rB}^2$

† $E_{rv} = \dfrac{\lambda_{rv}^* - \lambda_{rv}}{\lambda_{rv}}$, the relative error in vibrational eigenvalues.

$E_{rB} = \dfrac{\lambda_{rB}^* - \lambda_{rB}}{\lambda_{rB}}$, the relative error in buckling eigenvalues.

(b) *Buckling*

The governing equations are

$$-12w_{j-1} + 24w_j - 12w_{j+1} + 6\varepsilon(\theta_{j+1} - \theta_{j-1}) = \lambda\varepsilon^2(2w_j - w_{j-1} - w_{j+1})$$

$$6\varepsilon(w_{j-1} - w_{j+1}) + 2\varepsilon^2(\theta_{j-1} + 4\theta_j + \theta_{j+1}) = 0$$

where $\lambda = \dfrac{N}{EI}$.

Assuming the same solution as in equation (45) and following the same procedure as in the vibration case, we obtain

$$E_{rB} = \frac{\lambda_{rB}^* - \lambda_{rB}}{\lambda_{rB}} = \tfrac{1}{12}\varepsilon^2\lambda_{rB} + 0(\varepsilon^4\lambda_{rB}^2) \tag{47}$$

where $\lambda_{rB} = \left(\dfrac{r\pi}{n\varepsilon}\right)^2$ is the exact solution.

Errors for other cases can be calculated likewise and the results are summarized in Table 1.

It is seen that in all seven cases, the finite element solutions for the vibration problem converge to the correct answer but with a different rate of convergence. This is also true for the buckling problem except in the first two cases for which no geometric stiffness matrix can be generated because of the assumption of constant displacement.

6. CONCLUSIONS AND REMARKS

(1) The introduction of the term $\sum_n \int_{\partial V_n} T_i(\tilde{u}_i - u_i)\, dS$ in the variational functional over all of the interelement boundaries as originally done in[17] for the potential energy provides us all flexibility to construct different finite-element models. But it is more important to recognize that this provides us a rational means to match different kinds of elements. One may view that $\tilde{\boldsymbol{u}}$ is the displacement field of the adjacent elements at the boundaries common to the element being considered. By proper choice of $\boldsymbol{T}$ at the element boundary, we can obtain a reduced element stiffness matrix such as in[24] to match those of the neighboring elements. For example, if we want to match a linear strain triangle to constant strain triangles, we may visualize that over the edge of the linear strain triangle to be matched with the constant strain triangle, $\tilde{u}_i$ is linear while u_i is quadratic along that edge. By properly choosing T_i, for the linear strain triangle we can express the unknowns of u_i in terms of those of $\tilde{u}_i$.

(2) In the beam example discussed previously, the particular moment solution M^F has been assumed to be zero. In fact, a non-zero particular solution $M^F = \beta x^2$ has been tried for cases 1–7 and it is found that for cases 1–6, the resulting stiffness matrices $\boldsymbol{k}_1$ are unreasonably large and their corresponding eigenvalue solutions do not converge to the correct values. However, for case 7 the second term of $\boldsymbol{k}_1$ in equation (28) is identically zero and thus the assumed M^F has no effect at all. Furthermore, it can be shown that any $M^F = \beta x^n$ with $n \geq 2$ will have no effect on $\boldsymbol{k}_1$ and their corresponding eigenvalue solutions converge to correct values. These numerical examples seem to indicate that either $\boldsymbol{\sigma}^F = 0$ or $\boldsymbol{u} = \tilde{\boldsymbol{u}}$ must be satisfied in deriving the rational mass and geometric stiffness matrices of a hybrid stress element. The condition $\boldsymbol{\sigma}^F = 0$ has been used successfully by many authors in carrying out frequency analyses of plate and shell structures[13, 25].

(3) The present formulation presents an alternative way to obtain the loading vector for the hybrid stress model other than that given in[3] and[4]. The method suggested in the present paper is perhaps the most logical one, if one uses only low order equilibrating stresses. For example, in the case of plate bending, if one uses only linear moments for the equilibrating part of the moment distribution, then the final solution will depend on how the particular solution is assumed in[3, 4], since any particular solution is at least quadratic in x and/or y. In the present formulation, the load is lumped to the node according to the assumed interior displacement. A similar approach can be used to derive the equivalent loading vector due to thermal strains.

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Абстракт — Проведен унифицированный подход к проблеме получения геометрической жесткости и матричного элемента смешанных моделей. Вариационная формулировка основана на измененном принципе Рейсснера. Численная проверка иллюстрируется простым примером пучка. Также рассмотрена взаимозаменяемость элементов по методу крайних элементов и альтернативный способ получения вектора нагрузки смешаннжх моделей.